On the Integral of Fractional Poisson Processes

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Abstract

In this paper we consider the Riemann–Liouville fractional integral $\mathcal{N}^{\alpha,\nu}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N^\nu(s)\,\mathrm{d}s$, where $N^\nu(t)$, $t \geq 0$, is a fractional Poisson process of order $\nu \in (0,1]$, and $\alpha > 0$. We give the explicit bivariate distribution $\Pr\{N^\nu(s) = k, N^\nu(t) = r\}$, for $t \geq s$, $r \geq k$, the mean $\mathbb{E}\mathcal{N}^{\alpha,\nu}(t)$ and the variance $\mathbb{V}\mathrm{ar}\,\mathcal{N}^{\alpha,\nu}(t)$. We study the process $\mathcal{N}^{\alpha,1}(t)$ for which we are able to produce explicit results for the conditional and absolute variances and means. Much more involved results on $\mathcal{N}^{1,1}(t)$ are presented in the last section where also distributional properties of the integrated Poisson process (including the representation as random sums) is derived. The integral of powers of the Poisson process is examined and its connections with generalised harmonic numbers is discussed.

Keywords: Mittag–Leffler generalised functions; Riemann–Liouville fractional integrals; Skellam distribution.

1 Introduction

The fractional Poisson process $N^\nu(t)$, $t \geq 0$, $0 < \nu \leq 1$, has been introduced and studied in the last decade by Laskin [2003], Mainardi et al. [2004], Beghin and Orsingher [2009], Politi et al. [2011]. The starting point of the investigations of some authors was the derivation of the distribution

$$p_k^\nu(t) = \Pr\{N^\nu(t) = k\}, \qquad k \geq 0, \tag{1.1}$$

by solving the fractional equations

$$\begin{cases} \frac{\mathrm{d}^\nu}{\mathrm{d}t^\nu} p_k^\nu(t) = -\lambda p_k^\nu(t) + \lambda p_{k-1}^\nu(t), \\ p_k^\nu(0) = \begin{cases} 1, & k = 0, \\ 0, & k \geq 1. \end{cases} \end{cases} \tag{1.2}$$

The derivative appearing in (1.2) is meant in the sense of Riemann–Liouville in Laskin [2003] and in the sense of Dzhrbashyan–Caputo in Beghin and Orsingher [2009]. The distribution (1.1) reads

$$p_k^\nu(t) = \sum_{r=k}^\infty (-1)^{r-k} \binom{r}{k} \frac{(\lambda t^\nu)^r}{\Gamma(\nu r + 1)} = \frac{(\lambda t^\nu)^k}{k!} \sum_{r=0}^\infty \frac{(r+k)!}{r!} \frac{(-\lambda t^\nu)^r}{\Gamma(\nu(k+r)+1)}. \tag{1.3}$$

The fractional Poisson process is also constructed as a renewal process in Mainardi et al. [2004] and Beghin and Orsingher [2009] where is shown that its distribution coincides with (1.3). Meerschaert et al. [2011] treat and analyze in a unified way the process obtained from the time-fractional equation and the related renewal process with Mittag–Leffler distributed interarrival times. Furthermore, other generalizations in a fractional sense or as a renewal process with generalized Mittag–Leffler waiting times have been introduced by Orsingher and Polito [2012] and Cahoy and Polito [2013].

The intertime T_1^ν between successive events has distribution

$$\Pr\{T_1^\nu \in \mathrm{d}s\} = \lambda s^{\nu-1} E_{\nu,\nu}(-\lambda s^\nu)\,\mathrm{d}s, \qquad s \geq 0, \tag{1.4}$$

while the waiting time for the k-th event T_k^ν has distribution

$$\Pr\{T_k^\nu \in \mathrm{d}s\} = \lambda^k s^{\nu k-1} E_{\nu,\nu k}^k(-\lambda s^\nu)\,\mathrm{d}s, \qquad s \geq 0, \tag{1.5}$$

where

$$E_{\alpha,\eta}^\gamma(z) = \sum_{r=0}^\infty \frac{(\gamma)_r z^r}{r!\Gamma(\alpha r+\eta)}, \quad \alpha,\eta,\gamma \in \mathbb{C},\ \Re(\alpha),\Re(\eta),\Re(\gamma) > 0,\ z \in \mathbb{C}, \tag{1.6}$$

is the generalised Mittag–Leffler function [Haubold et al., 2011]. Note that $(\gamma)_r = \gamma(\gamma+1)\dots(\gamma+r-1)$, $\gamma \neq 0$, and that $E_{\alpha,\eta}(z) = E_{\alpha,\eta}^1(z)$.

The multivariate distribution of the fractional Poisson process $\Pr\{N^\nu(t_1) = n_1, \dots, N^\nu(t_k) = n_k\}$, where $t_1 < t_2 < \dots < t_k$, $n_1 \leq n_2 \leq \dots \leq n_k$, can be written down by considering its renewal structure and by exploiting (1.5) and (1.4) (see e.g. Politi et al. [2011]). We are able to give the explicite bivariate distribution in terms of generalised Mittag–Leffler functions. This plays a crucial role in the analysis of the variance of the fractional integral of the fractional Poisson process, i.e.

$$\mathcal{N}^{\alpha,\nu}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N^\nu(s)\,\mathrm{d}s, \qquad t \geq 0,\ 0 < \nu \leq 1,\ \alpha > 0, \tag{1.7}$$

with $\mathbb{E}\mathcal{N}^{\alpha,\nu}(t) = \lambda t^{\alpha+\nu}/\Gamma(\alpha+\nu+1)$.

For $\nu = 1$, we obtain the fractional integral of the classical Poisson process with intensity λ. The motivation in studying the above process is based on the fact that integrated non-negative stochastic processes and in general integrated counting processes often arise in the applied mathematical literature (see for example Jerwood [1970], Downton [1972], Hernández-Suárez and Castillo-Chavez [1999], Stefanov and Wang [2000], Pollett [2003], and the references therein). The analysis of the integrated process (1.7) is interesting as it permits to generalize the ideas behind such studies to a non-integer framework. Note also that for $\alpha \in \mathbb{N}$ the Riemann–Liouville fractional integral coincides with a classical multiple integral.

The main result for the Riemann–Liouville integral is the conditional second moment

$$\mathbb{E}\left\{\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} N(s)\,\mathrm{d}s\right)^2 \middle| N(t) = n\right\} = \frac{2nt^{2\alpha}\Gamma(2\alpha)}{\alpha\Gamma^2(\alpha)\Gamma(2\alpha+2)} + \frac{n(n-1)t^{2\alpha}}{\Gamma^2(\alpha+2)}. \tag{1.8}$$

Of course we have that [Kingman, 1993, page 21]

$$\mathbb{E}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} N(s)\,\mathrm{d}s \middle| N(t) = n\right\} = \frac{nt^\alpha}{\Gamma(\alpha+2)}, \tag{1.9}$$

and thus

$$\mathbb{E}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} N(s)\,\mathrm{d}s\right\} = \frac{\lambda t^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{1.10}$$

In light of (1.8) and (1.9) we are able to give the conditional variance of the fractional integral of the Poisson process:

$$\mathbb{V}\mathrm{ar}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} N(s)\,\mathrm{d}s \middle| N(t) = n\right\} = \frac{nt^{2\alpha}\alpha^2}{(2\alpha+1)\Gamma^2(\alpha+2)}. \tag{1.11}$$

Therefore we extract from (1.11) and (1.9) that

$$\mathbb{V}\mathrm{ar}\left(\mathcal{N}^{\alpha,1}(t)\right) = \frac{\lambda t^{2\alpha+1}}{(2\alpha+1)\Gamma^2(\alpha+1)}. \tag{1.12}$$

For $\alpha = 1$ we have the integral of the classical Poisson process which can be written as a random sum, i.e.

$$\int_0^t N(s)\,\mathrm{d}s \overset{\mathrm{d}}{=} \sum_{j=1}^{N(t)} X_j. \tag{1.13}$$

The random variables X_js appearing in (1.13) are i.i.d. with uniform law in $(0, t)$ independent of $N(t)$. In (1.13) we consider that the sum on the right hand side is empty for $N(t) = 0$. For the conditional integral of the Poisson process we have that

$$\mathbb{E}\left\{\int_0^t N(s)\,\mathrm{d}s \middle| N(t) = n\right\} = \frac{nt}{2}, \qquad \mathbb{V}\mathrm{ar}\left\{\int_0^t N(s)\,\mathrm{d}s \middle| N(t) = n\right\} = \frac{nt^2}{12}, \tag{1.14}$$

which are also special cases of (1.9) and (1.11) for $\alpha = 1$. The results (1.14) are also obtained by a different, alternative method.

Finally we examine in Section 4.1 the process $\mathring{N}(t) = N_\lambda(t) - N_\beta(t)$, $t \geq 0$, where $N_\lambda(t)$ and $N_\beta(t)$ are independent Poisson processes of parameter $\lambda > 0$ and $\beta > 0$, respectively. It is well-known that

$$\Pr\{\mathring{N}(t) = r\} = e^{-(\beta+\lambda)t}\left(\frac{\lambda}{\beta}\right)^{r/2} I_{|r|}(2t\sqrt{\lambda\beta}), \qquad r \in \mathbb{Z},\ t \geq 0, \tag{1.15}$$

where

$$I_\xi(z) = \sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^{2k+\xi}\frac{1}{k!\Gamma(k+\xi+1)} \tag{1.16}$$

is the modified Bessel function of the first kind. The distribution (1.15) is called the Skellam distribution. For the integral process

$$\int_0^t \mathring{N}(s)\,\mathrm{d}s = \int_0^t N_\lambda(s)\,\mathrm{d}s - \int_0^t N_\beta(s)\,\mathrm{d}s, \tag{1.17}$$

we show that

$$\int_0^t \mathring{N}(s)\,\mathrm{d}s \overset{\mathrm{d}}{=} \sum_{j=1}^{\widetilde{N}(t)} Z_j, \tag{1.18}$$

where $\widetilde{N}(t)$, $t \geq 0$, is a Poisson process with rate $\lambda + \beta$ and the random variables Z_js are i.i.d. with density

$$f(s) = \begin{cases} \frac{\beta}{t(\lambda+\beta)}, & -t < s \leq 0, \\ \frac{\lambda}{t(\lambda+\beta)}, & 0 < s < t. \end{cases} \tag{1.19}$$

Clearly, for $\beta = \lambda$, (1.19) becomes the uniform distribution in $(-t, t)$. As before, in (1.18) the sum is considered equal to zero when $\widetilde{N}(t) = 0$. We remark that integrals of different point processes have been considered over the years, for example in Puri [1966], where the integral of the birth and death process has been examined.

2 Fractional integral of the fractional Poisson process

For the fractional Poisson process $N^\nu(t)$, $t \geq 0$, described in the introduction we consider the Riemann–Liouville fractional integral

$$\mathcal{N}^{\alpha,\nu}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} N^\nu(s)\,\mathrm{d}s, \qquad t \geq 0,\ 0 < \nu \leq 1,\ \alpha > 0. \tag{2.1}$$

For integer values of α, say $\alpha = m$, the integral (2.1) can be written as

$$\frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} N^\nu(s)\,\mathrm{d}s = \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \dots \int_{s_{m-1}}^t N^\nu(s_m)\,\mathrm{d}s_m, \tag{2.2}$$

and this offers an intuitive interpretation of (2.1). By taking into account formula (2.7) of Beghin and Orsingher [2009] it is a quick matter to check that

$$\mathbb{E}\mathcal{N}^{\alpha,\nu}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbb{E}N^\nu(s)\,\mathrm{d}s = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\lambda s^\nu}{\Gamma(\nu+1)}\,\mathrm{d}s = \frac{\lambda t^{\alpha+\nu}}{\Gamma(\alpha+\nu+1)}. \tag{2.3}$$

Note that if $0 < \alpha+\nu \leq 1$, then $\mathbb{E}\mathcal{N}^{\alpha,\nu}(t) = \mathbb{E}N^{\alpha+\nu}(t)$.

The fractional Poisson process can be seen as a renewal process with intertime between successive events possessing distribution

$$\Pr\{T_1^\nu \in \mathrm{d}s\} = \lambda s^{\nu-1} E_{\nu,\nu}(-\lambda s^\nu)\,\mathrm{d}s, \qquad s \geq 0,\ 0 < \nu \leq 1. \tag{2.4}$$

This has been proved in Mainardi et al. [2004], Beghin and Orsingher [2009], and Politi et al. [2011]. The random instant of the occurrence of the kth event for $N^\nu(t)$, $t \geq 0$, is denoted by $T_k^\nu = \inf\{t\colon N^\nu(t) = k\}$. We need also the symbol $\mathcal{T}_h^{\nu,k} = T_{k+h}^\nu - T_k^\nu \overset{\mathrm{d}}{=} T_h^\nu$, where $\mathcal{T}_h^{\nu,k}$ represents the length of the time interval separating the kth and the $(k+h)$th event. The distribution of T_k^ν is given in Beghin and Orsingher [2010] as

$$\Pr\{T_k^\nu \in \mathrm{d}s\} = \lambda^k s^{\nu k-1} E_{\nu,\nu k}^k(-\lambda s^\nu)\,\mathrm{d}s, \qquad s \geq 0,\ 0 < \nu \leq 1. \tag{2.5}$$

Theorem 2.1. *The bivariate distribution of the fractional Poisson process reads*

$$\begin{aligned} &\Pr\{N^\nu(s) = k, N^\nu(t) = r\} \\ &= \lambda^r \int_0^s w^{\nu k-1} E_{\nu,\nu k}^k(-\lambda w^\nu)\,\mathrm{d}w \int_{s-w}^{t-w} y^{\nu-1} E_{\nu,\nu}(-\lambda y^\nu)(t-w-y)^{\nu(r-k-1)} E_{\nu,\nu(r-k-1)+1}^{r-k}(-\lambda(t-w-y)^\nu)\,\mathrm{d}y, \end{aligned} \tag{2.6}$$

where

$$E_{\alpha,\eta}^\gamma(z) = \sum_{r=0}^\infty \frac{\Gamma(\gamma+r) z^r}{r!\Gamma(\gamma)\Gamma(\alpha r+\eta)}, \qquad \alpha,\eta,\gamma \in \mathbb{C},\ \Re(\alpha),\Re(\eta),\Re(\gamma) > 0,\ z \in \mathbb{C}, \tag{2.7}$$

is the generalised Mittag–Leffler function.

Proof. In order to obtain the distribution (2.6) we must have a look at Figure 1, where the instants of occurrence of the relevant events are depicted. The bivariate distribution can be written as

$$\Pr\{N^\nu(s) = k, N^\nu(t) = r\} = \iiiint_D \Pr\{T_k^\nu \in \mathrm{d}w, \mathcal{T}_1^{\nu,k} \in \mathrm{d}y, \mathcal{T}_{r-k-1}^{\nu,k+1} \in \mathrm{d}\xi, \mathcal{T}_1^{\nu,r} \in \mathrm{d}\eta\}, \tag{2.8}$$

where $D = \{(0 < w < s) \cap (y+w > s) \cap (t > y+w+\xi > s) \cap (y+w+\xi+\eta) > t\}$. We have that (by keeping in mind the independence of the intertimes between events)

$$\Pr\{N^\nu(s) = k, N^\nu(t) = r\} \tag{2.9}$$

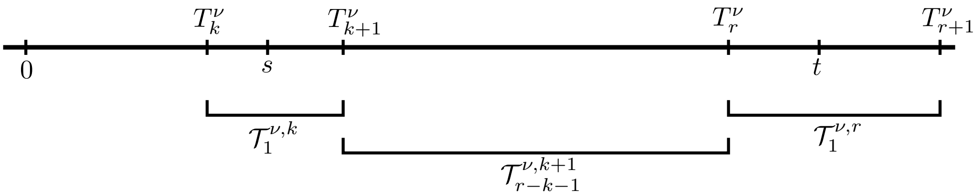

Figure 1: The instants of occurrence of the events and the related waiting times.

$$
\begin{aligned}
&= \int_0^s \int_{s-w}^{t-w} \int_0^{t-(w+y)} \int_{t-(y+w+\xi)}^{\infty} \Pr\{T_k^\nu \in \mathrm{d}w, \mathcal{T}_1^{\nu,k} \in \mathrm{d}y, \mathcal{T}_{r-k-1}^{\nu,k+1} \in \mathrm{d}\xi, \mathcal{T}_1^{\nu,r} \in \mathrm{d}\eta\} \\
&= \int_0^s \Pr\{T_k^\nu \in \mathrm{d}w\} \int_{s-w}^{t-w} \Pr\{\mathcal{T}_1^{\nu,k} \in \mathrm{d}y\} \int_0^{t-(w+y)} \Pr\{\mathcal{T}_{r-k-1}^{\nu,k+1} \in \mathrm{d}\xi\} \int_{t-(y+w+\xi)}^{\infty} \Pr\{\mathcal{T}_1^{\nu,r} \in \mathrm{d}\eta\} \\
&= \int_0^s \lambda^k w^{\nu k-1} E_{\nu,\nu k}^k(-\lambda w^\nu) \int_{s-w}^{t-w} \lambda y^{\nu-1} E_{\nu,\nu}(-\lambda y^\nu) \int_0^{t-(w+y)} \lambda^{r-k-1} \xi^{\nu(r-k-1)-1} E_{\nu,\nu(r-k-1)}^{r-k-1}(-\lambda \xi^\nu) \\
&\quad \times E_{\nu,1}(-\lambda(t-y-w-\xi)^\nu)\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}w.
\end{aligned}
$$

Writing down the integral in (2.9) consider Figure 1 and the independence of the intertimes T_k^ν, $\mathcal{T}_1^{\nu,k}$, $\mathcal{T}_{r-k-1}^{\nu,k+1}$ and $\mathcal{T}_1^{\nu,r}$ (with distributions (2.4) and (2.5)).

Formula (2.9) can be further simplified by recurring to the following relation (see e.g. Haubold et al. [2011], formula (11.7), page 17):

$$
\int_0^x (x-t)^{\beta-1} E_{\alpha,\beta}^\gamma[a(x-t)^\alpha] t^{\zeta-1} E_{\alpha,\zeta}^\sigma(at^\alpha)\mathrm{d}t = x^{\beta+\zeta-1} E_{\alpha,\beta+\zeta}^{\gamma+\sigma}(ax^\alpha), \tag{2.10}
$$

where $\alpha, \beta, \gamma, a, \zeta, \sigma \in \mathbb{C}$, and $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\zeta) > 0$, $\Re(\sigma) > 0$. With the necessary substitutions in (2.10), that is, $x = t - (w+y)$, $t = \xi$, $\alpha = \nu$, $\zeta = \nu(r-k-1)$, $\sigma = r-k-1$, $a = -\lambda$, and $\beta = 1$, we have that

$$
\begin{aligned}
&\Pr\{N^\nu(s) = k, N^\nu(t) = r\} \qquad (2.11) \\
&= \lambda^r \int_0^s w^{\nu k-1} E_{\nu,\nu k}^k(-\lambda w^\nu) \int_{s-w}^{t-w} y^{\nu-1} E_{\nu,\nu}(-\lambda y^\nu)(t-w-y)^{\nu(r-k-1)} E_{\nu,\nu(r-k-1)+1}^{r-k}(-\lambda(t-w-y)^\nu)\,\mathrm{d}y\,\mathrm{d}w.
\end{aligned}
$$

□

Remark 2.1. *We show now that* (2.6)*, for* $\nu = 1$*, that is for the classical homogenous Poisson process, yields*

$$
\Pr\{N^1(s) = k, N^1(t) = r\} = \frac{\lambda^r s^k (t-s)^{r-k}}{r!} \binom{r}{k} e^{-\lambda t}. \tag{2.12}
$$

Since $E_{1,1}(x) = e^x$ *and* $E_{1,k}^k(x) = e^x/(k-1)!$ *we have that*

$$
\Pr\{N^1(s) = k, N^1(t) = r\} = \lambda^r \frac{s^k}{k!} \frac{(t-s)^{r-k}}{(r-k)!} e^{-\lambda t}. \tag{2.13}
$$

Remark 2.2. *If we change the variable in the outer integral of* (2.6) *we get*

$$
\begin{aligned}
&\Pr\{N^\nu(s) = k, N^\nu(t) = r\} \qquad (2.14) \\
&= \lambda^r \int_{t-s}^t (t-z)^{\nu k-1} E_{\nu,\nu k}^k(-\lambda(t-z)^\nu)\,\mathrm{d}z \int_{z+s-t}^z y^{\nu-1} E_{\nu,\nu}(-\lambda y^\nu)(z-y)^{\nu(r-k-1)} E_{\nu,\nu(r-k-1)+1}^{r-k}(-\lambda(z-y)^\nu)\,\mathrm{d}y.
\end{aligned}
$$

In (2.14) *we have an integral of the form*

$$
\left(\mathbf{E}_{\rho,\mu,\omega;a+}^\gamma \phi\right)(x) = \int_a^x (x-t)^{\mu-1} E_{\rho,\mu}^\gamma\left(\omega(x-t)^\rho\right)\phi(t)\,\mathrm{d}t, \qquad x > a, \tag{2.15}
$$

treated in Prabhakar [1971], Saigo et al. [2004], Kilbas et al. [2002], Srivastava and Tomovski [2009]. The integral in (2.15) *is a generalisation of the Riemann–Liouville fractional integral.*

In light of (2.15)*, the bivariate distribution* (2.14) *can be written as*

$$\begin{aligned}
&\Pr\{N^\nu(s)=k, N^\nu(t)=r\} \qquad (2.16)\\
&= \lambda^r \int_{t-s}^t (t-z)^{\nu k-1} E^k_{\nu,\nu k}\left(-\lambda(t-z)^\nu\right)\left(\mathbf{E}^{r-k}_{\nu,\nu(r-k-1)+1,-\lambda;(z+s-t)+} y^{\nu-1}E_{\nu,\nu}(-\lambda y^\nu)\right)(z)\,\mathrm{d}z\\
&= \lambda^r \left(\mathbf{E}^k_{\nu,\nu k,-\lambda;(t-s)+}\left(\mathbf{E}^{r-k}_{\nu,\nu(r-k-1)+1,-\lambda;(z+s-t)+} y^{\nu-1}E_{\nu,\nu}(-\lambda y^\nu)\right)(z)\right)(t).
\end{aligned}$$

Remark 2.3. *For the second-order moment of the fractional integral we have that*

$$\mathbb{E}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}N^\nu(u)\,\mathrm{d}u\right\}^2 = \frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-u)^{\alpha-1}(t-v)^{\alpha-1}\mathbb{E}\{N^\nu(u)N^\nu(v)\}\,\mathrm{d}u\,\mathrm{d}w, \qquad (2.17)$$

where $\mathbb{E}\{N^\nu(u)N^\nu(v)\} = \sum_{k=0}^\infty\sum_{r=k}^\infty k\,r\Pr\{N^\nu(u)=k, N^\nu(v)=r\}$*,* $v>u$*. Unfortunately the complicated structure of the distribution* (2.6) *does not permit us to determine a closed form for* (2.17)*.*

3 Fractional integral of the homogenous Poisson process

We now restrict the analysis of (2.1) to the case $\nu=1$, that is we study the Riemann–Liouville fractional integral of the homogenous Poisson process:

$$\mathcal{N}^{\alpha,1}(t) = \mathcal{N}^\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}N(s)\,\mathrm{d}s, \qquad (3.1)$$

for $\alpha>0$, $t\geq 0$. From (2.3) we have that

$$\mathbb{E}\mathcal{N}^\alpha(t) = \frac{\lambda t^{\alpha+1}}{\Gamma(\alpha+2)}. \qquad (3.2)$$

We are now in the position of evaluating explicitly $\mathbb{E}\left[\mathcal{N}^\alpha(t)\right]^2$ and also $\mathbb{E}\left\{\left[\mathcal{N}^\alpha(t)\right]^2|N(t)=n\right\}$. Thus we can state the following theorem.

Theorem 3.1. *The variance of* (3.1) *has the following form:*

$$\mathbb{V}ar\mathcal{N}^\alpha(t) = \frac{\lambda t^{2\alpha+1}}{(2\alpha+1)\Gamma^2(\alpha+1)}, \qquad t\geq 0,\ \alpha>0. \qquad (3.3)$$

Proof. We start by evaluating $\mathbb{E}\left[\mathcal{N}^\alpha(t)\right]^2$.

$$\begin{aligned}
\mathbb{E}\left[\mathcal{N}^\alpha(t)\right]^2 &= \frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-s)^{\alpha-1}(t-w)^{\alpha-1}\mathbb{E}\left[N(s)N(w)\right]\,\mathrm{d}s\,\mathrm{d}w \qquad (3.4)\\
&= \frac{2}{\Gamma^2(\alpha)}\int_0^t\int_s^t (t-s)^{\alpha-1}(t-w)^{\alpha-1}(\lambda s+\lambda^2 sw)\,\mathrm{d}w\,\mathrm{d}s\\
&= \frac{2}{\Gamma^2(\alpha)}\left\{\lambda\int_0^t s(t-s)^{\alpha-1}\mathrm{d}s\int_s^t (t-w)^{\alpha-1}\mathrm{d}w + \lambda^2\int_0^t s(t-s)^{\alpha-1}\mathrm{d}s\int_s^t w(t-w)^{\alpha-1}\mathrm{d}w\right\}\\
&= \frac{2}{\Gamma^2(\alpha)}\left\{\frac{\lambda}{\alpha}\int_0^t s(t-s)^{2\alpha-1}\mathrm{d}s + \frac{\lambda^2}{\alpha}\int_0^t s^2(t-s)^{2\alpha-1}\mathrm{d}s + \frac{\lambda^2}{\alpha(\alpha+1)}\int_0^t s(t-s)^{2\alpha}\mathrm{d}s\right\}\\
&= \frac{2}{\Gamma^2(\alpha)}\left\{\frac{\lambda t^{2\alpha+1}}{2\alpha^2(2\alpha+1)} + \frac{\lambda^2 t^{2\alpha+2}}{2\alpha(\alpha+1)^2(2\alpha+1)} + \frac{\lambda^2 t^{2\alpha+2}}{2\alpha^2(2\alpha+1)(\alpha+1)}\right\}\\
&= \frac{1}{\Gamma^2(\alpha)}\left\{\frac{\lambda t^{2\alpha+1}}{\alpha^2(2\alpha+1)} + \frac{\lambda^2 t^{2\alpha+2}}{\alpha^2(\alpha+1)^2}\right\} = \frac{\lambda t^{2\alpha+1}}{(2\alpha+1)\Gamma^2(\alpha+1)} + \frac{\lambda^2 t^{2\alpha+2}}{\Gamma^2(\alpha+2)}.
\end{aligned}$$

By considering (3.2) we immediately arrive at the claimed result. □

Remark 3.1. *For the conditional mean, we directly arrive at the result*

$$\mathbb{E}\left(\mathcal{N}^\alpha(t)\big|N(t)=n\right) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{ns}{t}\mathrm{d}s = \frac{nt^\alpha}{\Gamma(\alpha+2)}. \tag{3.5}$$

In (3.5) *we considered that*

$$\Pr\{N(s)=r|N(t)=n\} = \binom{n}{r}\left(\frac{s}{t}\right)^r\left(1-\frac{s}{t}\right)^{n-r}, \qquad 0\le r\le n,\ s<t. \tag{3.6}$$

In order to obtain the conditional variance of $\mathcal{N}^\alpha(t)$ we need the following result.

Theorem 3.2. *For the homogenous Poisson process we have that*

$$\mathbb{E}\left\{N(s)N(w)\big|N(t)=n\right\} = \frac{ns}{t} + n(n-1)\frac{sw}{t^2}, \qquad 0<s<w<t. \tag{3.7}$$

Proof. In order to obtain (3.7) we evaluate the following bivariate conditional distribution. For $s<w<t$ we have that

$$\Pr\{N(s)=h, N(w)=k|N(t)=n\} = \Pr\{N(s)=h|N(w)=k\}\Pr\{N(w)=k|N(t)=n\} \tag{3.8}$$

because the time-reversed Poisson process is Markovian. From (3.7) we obtain the following trinomial distribution.

$$\begin{aligned}&\Pr\{N(s)=h, N(w)=k|N(t)=n\} \qquad (3.9)\\ &= \binom{k}{h}\left(\frac{s}{w}\right)^h\left(1-\frac{s}{w}\right)^{k-h}\binom{n}{k}\left(\frac{w}{t}\right)^k\left(1-\frac{w}{t}\right)^{n-k} = \frac{n!s^h(w-s)^{k-h}(t-w)^{n-k}}{h!(k-h)!(n-k)!t^n}, \qquad 0<s<w<t.\end{aligned}$$

We evaluate directly the conditional mixed moment of (3.9) as follows.

$$\begin{aligned}&\mathbb{E}\{N(s)N(w)|N(t)=n\} = \sum_{h=0}^n\sum_{k=h}^n k\,h\frac{n!s^h(w-s)^{k-h}(t-w)^{n-k}}{h!(k-h)!(n-k)!t^n} \qquad (3.10)\\
&= \frac{n!}{t^n}\sum_{h=1}^n\frac{s^h}{(h-1)!}\sum_{r=0}^{n-h}\frac{(h+r)}{r!(n-r-h)!}(w-s)^r(t-w)^{n-r-h}\\
&= \frac{n!}{t^n}\sum_{h=1}^n\frac{s^h}{(h-1)!}\left[\frac{h}{(n-h)!}\sum_{r=0}^{n-h}\binom{n-h}{r}(w-s)^r(t-w)^{n-r-h}\right.\\
&\quad\left.+\frac{(w-s)}{(n-h-1)!}\sum_{l=0}^{n-h-1}\binom{n-h-1}{l}(w-s)^l(t-w)^{n-h-1-l}\right]\\
&= \frac{n!}{t^n}\sum_{h=1}^n\frac{s^h}{(h-1)!}\left[\frac{h}{(n-h)!}(t-s)^{n-h}+\frac{(w-s)}{(n-h-1)!}(t-s)^{n-1-h}\right]\\
&= \frac{n!}{t^n}\left[\sum_{m=0}^{n-1}\binom{n-1}{m}s^{m+1}(t-s)^{n-1-m}\frac{m+1}{(n-1)!}+\sum_{h=1}^{n-1}\frac{s^h}{(h-1)!}\frac{(w-s)}{(n-h-1)!}(t-s)^{n-1-h}\right]\\
&= \frac{n!}{t^n}\left[\frac{1}{(n-1)!}\left(t^{n-1}(n-1)\frac{s^2}{t}+t^{n-1}s\right)+\frac{(w-s)st^{n-2}}{(n-2)!}\right]\\
&= n(n-1)\frac{s^2}{t^2}+\frac{ns}{t}+(w-s)\frac{s(n-1)n}{t^2} = \frac{n(n-1)}{t^2}ws+\frac{ns}{t}.\end{aligned}$$

□

Remark 3.2. *As a simple check we note that*

$$\mathbb{E}\left[\mathbb{E}\left(N(s)N(w)|N(t)\right)\right] = \lambda^2ws+\lambda s, \qquad 0<s<t. \tag{3.11}$$

Since for $w = s$

$$\mathbb{E}\left([N(s)]^2|N(t)=n\right) = \frac{n(n-1)}{t^2}s^2 + \frac{ns}{t}, \tag{3.12}$$

the conditional variance reads

$$\mathbb{V}ar\,(N(s)|N(t)=n) = \frac{ns}{t} - \frac{ns^2}{t^2}. \tag{3.13}$$

In turn, the unconditional variance can be obtained as follows.

$$\begin{aligned}\mathbb{V}ar\,N(s) &= \mathbb{E}\left[\mathbb{V}ar\,(N(s)|N(t))\right] + \mathbb{V}ar\left[\mathbb{E}\left(N(s)|N(t)\right)\right] \\ &= \left(\frac{s}{t} - \frac{s^2}{t^2}\right)\mathbb{E}N(t) + \mathbb{V}ar\left(\frac{s}{t}N(t)\right) = \left(\frac{s}{t} - \frac{s^2}{t^2}\right)\lambda t + \frac{s^2}{t}\lambda = \lambda s.\end{aligned} \tag{3.14}$$

We arrive at the conditional variance of the Riemann–Liouville fractional integral of the Poisson process in the next theorem.

Theorem 3.3. *We have that*

$$\mathbb{V}ar\,(\mathcal{N}^\alpha(t)|N(t)=n) = \frac{nt^{2\alpha}\alpha^2}{(2\alpha+1)\Gamma^2(\alpha+2)}. \tag{3.15}$$

Proof. Exploiting result (3.10) we have

$$\begin{aligned}&\mathbb{E}\left\{\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}N(s)\mathrm{d}s\right)^2 \middle| N(t)=n\right\} \\ &= \frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^s (t-s)^{\alpha-1}(t-w)^{\alpha-1}\mathbb{E}\left(N(s)N(w)|\,N(t)=n\right)\mathrm{d}s\,\mathrm{d}w \\ &\quad + \frac{1}{\Gamma^2(\alpha)}\int_0^t\int_s^t (t-s)^{\alpha-1}(t-w)^{\alpha-1}\mathbb{E}\left(N(s)N(w)|\,N(t)=n\right)\mathrm{d}s\,\mathrm{d}w \\ &= \frac{n}{t\Gamma^2(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s\int_0^s w(t-w)^{\alpha-1}\,\mathrm{d}w + \frac{n}{t\Gamma^2(\alpha)}\int_0^t s(t-s)^{\alpha-1}\,\mathrm{d}s\int_s^t (t-w)^{\alpha-1}\,\mathrm{d}w \\ &\quad + \frac{n(n-1)}{t^2\Gamma^2(\alpha)}\int_0^t\int_0^t (t-s)^{\alpha-1}(t-w)^{\alpha-1}sw\,\mathrm{d}s\,\mathrm{d}w \\ &= \frac{2nt^{2\alpha}\Gamma(2\alpha)}{\alpha\Gamma^2(\alpha)\Gamma(2\alpha+2)} + \frac{n(n-1)t^{2\alpha}}{\Gamma^2(\alpha+2)}.\end{aligned} \tag{3.16}$$

From results (3.16) and (3.5), the conditional variance then simply reads

$$\mathbb{V}\mathrm{ar}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}N(s)\,\mathrm{d}s\middle| N(t)=n\right\} = \frac{2nt^{2\alpha}\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha+1)\Gamma(2\alpha+2)} - \frac{nt^{2\alpha}}{\Gamma^2(\alpha+2)} = \frac{nt^{2\alpha}\alpha^2}{(2\alpha+1)\Gamma^2(\alpha+2)}. \tag{3.17}$$

□

Remark 3.3. *The unconditional variance* (3.3) *can be easily retrieved as follows.*

$$\begin{aligned}&\mathbb{V}ar\,(\mathcal{N}^\alpha(t)) = \\ &\mathbb{E}\left\{\mathbb{V}ar\,(\mathcal{N}^\alpha(t)|N(t))\right\} + \mathbb{V}ar\left\{\mathbb{E}\left(\mathcal{N}^\alpha(t)|N(t)\right)\right\} = \frac{\lambda t^{2\alpha+1}\alpha^2}{(2\alpha+1)\Gamma^2(\alpha+2)} + \frac{\lambda t^{2\alpha+1}}{\Gamma^2(\alpha+2)} = \frac{\lambda t^{2\alpha+1}}{(2\alpha+1)\Gamma^2(\alpha+1)}.\end{aligned} \tag{3.18}$$

4 Integral of the homogenous Poisson process

For the integral of the Poisson process we have a representation in terms of random sums.

Theorem 4.1. *For the homogenous Poisson process $N(t)$, $t \geq 0$, we have that*

$$\mathcal{N}(t) = \int_0^t N(s)\,\mathrm{d}s \stackrel{d}{=} \sum_{j=1}^{N(t)} X_j = \mathfrak{N}(t), \tag{4.1}$$

where the X_js are i.i.d. random variables, uniform in $[0,t]$. In (4.1), the sum in the right hand side is intended to be equal to zero when $N(t) = 0$.

Proof. If $N(t) = n$, and $\tau_1, \tau_2, \dots, \tau_n$, are the random instants at which the Poisson events appear, we have that the integral of the Poisson $A_n(t)$, $t \geq 0$, reads

$$A_n(t) = \sum_{j=2}^n (\tau_j - \tau_{j-1})(j-1) + n(t - \tau_n). \tag{4.2}$$

Since

$$\Pr\{\tau_1 \in \mathrm{d}s_1, \dots, \tau_n \in \mathrm{d}s_n\} = \frac{n!}{t^n}, \qquad 0 < s_1 < s_2 < \dots < s_n < t, \tag{4.3}$$

we have

$$\mathbb{E}e^{i\beta A_n(t)} = \frac{n!}{t^n} \int_0^t \mathrm{d}s_1 \dots \int_{s_{n-1}}^t \mathrm{d}s_n e^{i\beta\left[\sum_{j=2}^n (s_j - s_{j-1})(j-1) + n(t-s_n)\right]} = \frac{n!}{t^n} F_n(\beta, t). \tag{4.4}$$

It is evident that the functions $F_n(\beta, t)$ satisfy the equations

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} F_n(\beta, t) = i\,n\,\beta F_n(\beta, t) + F_{n-1}(\beta, t), & n \geq 1, \\ F_n(\beta, 0) = 0, \end{cases} \tag{4.5}$$

where $F_0(\beta, t) = 0$. Now we show by induction that $F_n(\beta, t) = \left(e^{i\beta t} - 1\right)^n / (n!(i\beta)^n)$, $n \geq 0$. From (4.5) we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} F_n(\beta, t) - i\,n\,\beta F_n(\beta, t) = \frac{\left(e^{i\beta t} - 1\right)^{n-1}}{(n-1)!\,(i\beta)^{n-1}}, \tag{4.6}$$

and the solutions turn out to be

$$F_n(\beta, t) = e^{in\beta t} \left[\int_0^t \frac{\left(e^{i\beta s} - 1\right)^{n-1}}{(n-1)!\,(i\beta)^{n-1}} e^{-in\beta s} \mathrm{d}s \right] = \frac{e^{in\beta t}}{(n-1)!\,(i\beta)^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^{n-1-m} \int_0^t e^{im\beta s - in\beta s} \mathrm{d}s \tag{4.7}$$

$$\begin{aligned}
&= \frac{e^{in\beta t}}{(n-1)!(i\beta)^{n-1}} \frac{1}{i\beta} \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^{n-1-m} \frac{\left(e^{-i\beta t(n-m)} - 1\right)}{(m-n)} \\
&= \frac{e^{in\beta t}}{(n-1)!(i\beta)^n} \sum_{m=0}^{n-1} \frac{(n-1)!}{m!(n-1-m)!(n-m)} (-1)^{n-m} \left[e^{-i\beta t(n-m)} - 1\right] \\
&= \frac{e^{i\beta nt}}{n!(i\beta)^n} \left[\sum_{r=0}^{n-1} (-1)^{n-r} \left(e^{-i\beta t(n-r)} - 1\right) \right] = \frac{e^{i\beta nt}}{n!(i\beta)^n} \left[\sum_{r=1}^n \binom{n}{r} (-1)^r \left(e^{-i\beta tr} - 1\right) \right] \\
&= \frac{e^{i\beta nt}}{n!(i\beta)^n} \left[\sum_{r=0}^n \binom{n}{r} (-1)^r \left(e^{-i\beta tr} - 1\right) \right] = \frac{e^{i\beta nt}\left(1 - e^{-i\beta t}\right)^n}{n!(i\beta)^n} = \frac{\left(e^{i\beta t} - 1\right)^n}{n!(i\beta)^n}.
\end{aligned}$$

The characteristic function of $A_n(t)$ can thus be written as

$$\mathbb{E}e^{i\beta A_n(t)} = \frac{\left(e^{i\beta t} - 1\right)^n}{t^n (i\beta)^n}, \tag{4.8}$$

so that

$$\mathbb{E}e^{i\beta\mathcal{N}(t)} = e^{-\lambda t}\sum_{n=0}^{\infty}\left(e^{i\beta t}-1\right)^n \frac{(\lambda t)^n}{t^n}\frac{1}{(i\beta)^n}\frac{1}{n!} = e^{-\lambda t+\frac{\lambda}{i\beta}\left(e^{i\beta t}-1\right)} = e^{\lambda\int_0^t\left(e^{i\beta s}-1\right)\mathrm{d}s}, \tag{4.9}$$

which is the characteristic function of the compound process $\mathfrak{N}(t)$, $t \geq 0$. □

Remark 4.1. *From* (4.9), *we have that*

$$\mathbb{E}e^{i\beta\mathcal{N}(t)} = e^{i\lambda\frac{\beta t^2}{2}-\lambda\frac{\beta^2t^3}{6}+o(t^3)}. \tag{4.10}$$

This shows that for small t the integrated Poisson process is Gaussian with mean $\lambda t^2/2$ and variance $\lambda t^3/3$. The parameters of the approximating Gaussian coincide with the mean and variance of $\mathcal{N}(t)$.

In the previous section we have obtained that

$$\mathbb{E}\left\{\mathcal{N}^\alpha(t)|N(t)=n\right\} = \frac{nt^\alpha}{\Gamma(\alpha+2)}, \qquad n \geq 0, \tag{4.11}$$

and thus, for $\alpha = 1$, $\mathbb{E}\left\{\mathcal{N}(t)|N(t)=n\right\} = nt/2$. We are able to derive this result with a different technique and, in the same way, to obtain

$$\mathbb{E}\left\{\int_0^t [N(s)]^k\,\mathrm{d}s\middle|N(t)=n\right\} \tag{4.12}$$

for $k = 2, 3$. The same technique is applied for the derivation of

$$\mathbb{V}\mathrm{ar}\left\{\int_0^t N(s)\,\mathrm{d}s\middle|N(t)=n\right\}. \tag{4.13}$$

Before stating the next theorem we recall again that the conditional distribution of $N(s)$, given $N(t) = n$, $s < t$, is Binomial$(n, s/t)$ (see formula (3.6) and Kingman [1993, page 21]).

Theorem 4.2. *For the integrated powers of the Poisson process we have that*

$$\mathbb{E}\left\{\int_0^t [N(s)]^k\,\mathrm{d}s\middle|N(t)=n\right\} = \frac{t}{n+1}\sum_{j=1}^n j^k. \tag{4.14}$$

Proof. For $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$, under the condition that $N(t) = n$, we can write

$$\int_0^t [N(s)]^k\,\mathrm{d}s = \sum_{j=1}^{n+1}\int_{t_{j-1}}^{t_j}[N(s)]^k\,\mathrm{d}s = \sum_{j=1}^{n+1}(j-1)^k(t_j-t_{j-1}). \tag{4.15}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left\{\int_0^t [N(s)]^k\,\mathrm{d}s\middle|N(t)=n\right\} &= \frac{n!}{t^n}\int_0^t \mathrm{d}t_1\ldots\int_{t_{j-1}}^t \mathrm{d}t_j\ldots\int_{t_{n-1}}^t \mathrm{d}t_n\sum_{j=1}^{n+1}(j-1)^k(t_j-t_{j-1}) \qquad (4.16)\\ &= \frac{n!}{t^n}\sum_{j=1}^{n+1}\frac{(j-1)^k}{(n-j)!}\int_0^t \mathrm{d}t_1\ldots\int_{t_{j-1}}^t(t_j-t_{j-1})(t-t_j)^{n-j}\mathrm{d}t_j.\end{aligned}$$

Since

$$\int_{t_{j-1}}^t (t_j-t_{j-1})(t-t_j)^{n-j}\mathrm{d}t_j = \int_0^{t-t_{j-1}} w^{n-j}(t-t_{j-1}-w)\,\mathrm{d}w \quad = (t-t_{j-1})^{n-j+2}\frac{\Gamma(2)\Gamma(n-j-1)}{\Gamma(n-j+3)}, \tag{4.17}$$

we arrive at

$$\mathbb{E}\left\{\int_0^t [N(s)]^k\,\mathrm{d}s\middle|N(t)=n\right\} = \frac{n!}{t^n}\sum_{j=1}^{n+1}\frac{(j-1)^k}{(n-j)!}\int_0^t \mathrm{d}t_1\ldots\int_{t_{j-2}}^t \mathrm{d}t_{j-1}(t-t_{j-1})^{n-j+2}\frac{(n-j)!}{(n-j+2)!} \tag{4.18}$$

$$= \frac{n!}{t^n} \sum_{j=1}^{n+1} \frac{(j-1)^k}{(n+1)!} = \frac{t}{n+1} \sum_{j=1}^{n} j^k.$$

□

Remark 4.2. *Explicit results can be given for small values of k:*

$$\mathbb{E}\left\{ \int_0^t [N(s)]^k \, \mathrm{d}s \middle| N(t) = n \right\} = \begin{cases} \frac{nt}{2}, & k = 1, \\ \frac{n(2n+1)t}{6}, & k = 2, \\ \frac{n^2(n+1)t}{4}, & k = 3. \end{cases} \tag{4.19}$$

The unconditional mean values have therefore the form

$$\mathbb{E}\left\{ \int_0^t [N(s)]^k \, \mathrm{d}s \right\} = \begin{cases} \frac{\lambda t}{2}, & k = 1, \\ \frac{\lambda^2 t^3}{3} + \frac{\lambda t^2}{2}, & k = 2, \\ \frac{\lambda^3 t^4}{4} + \lambda^2 t^3 + \frac{\lambda t^2}{2}, & k = 3. \end{cases} \tag{4.20}$$

By applying the same technique as in Theorem 4.2 we obtain the conditional variance.

Theorem 4.3. *We have the following explicit results.*

$$\mathbb{E}\left\{ \left[\int_0^t N(s) \, \mathrm{d}s \right]^2 \middle| N(t) = n \right\} = \frac{n(3n+1)t^2}{12}, \qquad \mathbb{V}ar\left\{ \int_0^t N(s) \, \mathrm{d}s \middle| N(t) = n \right\} = \frac{nt^2}{12}. \tag{4.21}$$

Proof. If we assume that $N(t) = n$, the following decomposition holds.

$$\left(\int_0^t N(s) \, \mathrm{d}s \right)^2 = \sum_{j=1}^{n+1} (j-1)^2 (t_j - t_{j-1})^2 + 2 \sum_{1 \le j < r \le n+1} (j-1)(r-1)(t_j - t_{j-1})(t_r - t_{r-1}), \tag{4.22}$$

for $0 = t_0 < t_1 < \cdots < t_j < \cdots < t_n < t_{n+1} = t$. Note that

$$\begin{aligned} &\frac{n!}{t^n} \int_0^t \mathrm{d}t_1 \dots \int_{t_{j-1}}^t (t_j - t_{j-1}) \, \mathrm{d}t_j \int_{t_j}^t \mathrm{d}t_{j+1} \dots \int_{t_{r-1}}^t (t_r - t_{r-1}) \, \mathrm{d}t_r \int_{t_r}^t \mathrm{d}t_{r+1} \dots \int_{t_{n-1}}^t \mathrm{d}t_n \\ &= \frac{n!}{t^n} \int_0^t \mathrm{d}t_1 \dots \int_{t_{j-1}}^t (t_j - t_{j-1}) \, \mathrm{d}t_j \int_{t_j}^t \mathrm{d}t_{j+1} \dots \int_{t_{r-1}}^t (t_r - t_{r-1}) \frac{(t-t_r)^{n-r}}{(n-r)!} \mathrm{d}t_r \\ &= \frac{n!}{t^n \Gamma(n-r+3)} \int_0^t \mathrm{d}t_1 \dots \int_{t_{j-1}}^t (t_j - t_{j-1}) \, \mathrm{d}t_j \int_{t_j}^t \mathrm{d}t_{j+1} \int_{t_{r-2}}^t \mathrm{d}t_{r-1} (t - t_{r-1})^{n-r+2} \\ &= \frac{n!}{t^n \Gamma(n-j+2)} \int_0^t \mathrm{d}t_1 \dots \int_{t_{j-1}}^t \mathrm{d}t_j (t-t_j)^{n-j+1} (t_j - t_{j-1}) \\ &= \frac{n!}{t^n \Gamma(n-j+2)} \int_0^t \mathrm{d}t_1 \dots \int_{t_{j-2}}^t \mathrm{d}t_{j-1} (t-t_{j-1})^{n-j+3} \frac{\Gamma(n-j+2)\Gamma(2)}{\Gamma(n-j+4)} \\ &= \frac{n!}{\Gamma(n-j+4)} \frac{t^2}{(n-j+4)(n-j+5)\dots(n+2)} = \frac{t^2}{(n+2)(n+1)}. \end{aligned} \tag{4.23}$$

In the same way we have that

$$\begin{aligned} &\frac{n!}{t^n} \int_0^t \mathrm{d}t_1 \dots \int_{t_{j-1}}^t (t_j - t_{j-1})^2 \mathrm{d}t_j \int_{t_j}^t \mathrm{d}t_{j+1} \dots \int_{t_{n-1}}^t \mathrm{d}t_n = \frac{n!}{t^n} \int_0^t \mathrm{d}t_1 \dots \int_{t_{j-1}}^t (t_j - t_{j-1})^2 \frac{(t-t_j)^{n-j}}{(n-j)!} \mathrm{d}t_j \\ &= \frac{n!}{t^n} \int_0^t \mathrm{d}t_1 \dots \int_{t_{j-2}}^t \mathrm{d}t_{j-1} (t - t_{j-1})^{n-j+3} \frac{2}{(n-j+3)!} = \frac{2n! t^{n+2}}{t^n (n+2)!} = \frac{2t^2}{(n+1)(n+2)}. \end{aligned} \tag{4.24}$$

In light of formulae (4.23), (4.24) and decomposition (4.22), we have

$$\begin{aligned}
&\mathbb{E}\left\{\left(\int_0^t N(s)\mathrm{d}s\right)^2\middle|N(t)=n\right\} = \frac{2t^2}{(n+1)(n+2)}\sum_{j=1}^{n+1}(j-1)^2+2\sum_{j=1}^{n}(j-1)\sum_{r=j+1}^{n+1}(r-1)\frac{t^2}{(n+1)(n+2)} \qquad (4.25)\\
&= \frac{2t^2}{(n+1)(n+2)}\left[\sum_{j=1}^{n} j^2+\sum_{j=1}^{n}(j-1)\sum_{r=j+1}^{n+1}(r-1)\right]\\
&= \frac{2t^2}{(n+1)(n+2)}\left[\frac{n(n+1)(2n+1)}{6}+\sum_{j=2}^{n}(j-1)\left(\frac{n(n+1)}{2}-\frac{j(j-1)}{2}\right)\right]\\
&= \frac{2t^2}{(n+1)(n+2)}\left[\frac{n(n+1)(2n+1)}{6}+\frac{n^2(n+1)(n-1)}{4}-\frac{1}{2}\left(\frac{n(n-1)}{2}\right)^2-\frac{1}{2}\frac{n(n-1)(2(n-1)+1)}{6}\right]\\
&= \frac{t^2}{(n+1)(n+2)}\left[\frac{n(n+1)(2n+1)}{3}-\frac{n(n-1)(2n-1)}{6}+\frac{n^2(n-1)(n+3)}{4}\right]\\
&= \frac{nt^2}{12(n+1)(n+2)}\left[4(n+1)(2n+1)-2(n-1)(2n-1)+3n(n-1)(n+3)\right]\\
&= \frac{nt^2}{12(n+2)}\left[4(2n+1)+(n-1)(3n+2)\right]=\frac{n(3n+1)t^2}{12}.
\end{aligned}$$

The conditional variance easily follows. ◻

Remark 4.3. *The results of Theorem 3.3, for $\alpha=1$, coincide with* (4.21).

We also observe that

$$\mathbb{V}ar\left\{\int_0^t N(s)\,\mathrm{d}s\right\}=\mathbb{E}\left\{\mathbb{V}ar\left(\int_0^t N(s)\,\mathrm{d}s\middle|N(t)\right)\right\}+\mathbb{V}ar\left\{\mathbb{E}\left(\int_0^t N(s)\,\mathrm{d}s\middle|N(t)\right)\right\}=\frac{\lambda t^3}{3}, \tag{4.26}$$

and this coincides with (3.18) *for $\alpha=1$.*

4.1 Properties of the integral of $\mathring{N}(t)=N_\lambda(t)-N_\beta(t)$

It is well-known that for two independent Poisson processes the difference $\mathring{N}(t)=N_\lambda(t)-N_\beta(t)$, $t\geq 0$, (which can be used, for example, in modelling immigration-emigration processes) has Skellam distribution:

$$\Pr\{\mathring{N}(t)=r\}=e^{-(\beta+\lambda)t}\left(\frac{\lambda}{\beta}\right)^{r/2}I_{|r|}(2t\sqrt{\lambda\beta}), \qquad r\in\mathbb{Z},\ t>0, \tag{4.27}$$

where

$$I_\alpha(z)=\sum_{m=0}^{\infty}\frac{(z/2)^{2m+\alpha}}{m!\Gamma(m+\alpha+1)} \tag{4.28}$$

is the modified Bessel function of the first kind.

For the integral of the difference of the two Poisson processes we have that

$$\begin{aligned}
&\mathbb{E}e^{i\mu\left(\int_0^t N_\lambda(s)\mathrm{d}s-\int_0^t N_\beta(s)\mathrm{d}s\right)}=e^{\lambda\int_0^t\left(e^{i\mu s}-1\right)\mathrm{d}s+\beta\int_0^t\left(e^{-i\mu s}-1\right)\mathrm{d}s}=e^{-(\lambda+\beta)t+\int_0^t\left(\lambda e^{i\mu s}+\beta e^{-i\mu s}\right)\mathrm{d}s} \qquad (4.29)\\
&=e^{-(\lambda+\beta)t+\frac{\lambda e^{i\mu t}-\lambda}{i\mu}-\frac{\beta e^{-i\mu t}-\beta}{i\mu}}=e^{-(\lambda+\beta)t+\frac{1}{i\mu}(\lambda-\beta)\cos\mu t+\frac{(\lambda+\beta)}{\mu}\sin\mu t-\frac{(\lambda-\beta)}{i\mu}}=e^{(\lambda+\beta)t\left(\frac{\sin\mu t}{\mu t}-1\right)}e^{-\frac{(\lambda-\beta)}{i\mu}(1-\cos\mu t)}.
\end{aligned}$$

Theorem 4.4. *For the difference of integrated Poisson processes we have that*

$$\int_0^t\mathring{N}(s)\mathrm{d}s=\int_0^t N_\lambda(s)\mathrm{d}s-\int_0^t N_\beta(s)\mathrm{d}s\overset{\mathrm{d}}{=}\sum_{j=1}^{\widetilde{N}(t)}Z_j, \tag{4.30}$$

where $\widetilde{N}(t)$, $t \geq 0$, is a Poisson process of rate $\lambda+\beta$ and the Z_js are i.i.d. random variables with law

$$f(s) = \begin{cases} \frac{\beta}{t(\lambda+\beta)}, & -t < s \leq 0, \\ \frac{\lambda}{t(\lambda+\beta)}, & 0 < s < t. \end{cases} \tag{4.31}$$

Proof. The claimed result can be proved by resorting to the characteristic function.

$$\mathbb{E}e^{i\mu\left[\int_0^t N_\lambda(s)\mathrm{d}s - \int_0^t N_\beta(s)\mathrm{d}s\right]} = e^{\lambda\int_0^t (e^{i\mu s}-1)\mathrm{d}s + \beta\int_0^t (e^{-i\mu s}-1)\mathrm{d}s} = e^{(\lambda+\beta)t\int_{-t}^t \left(e^{i\mu s}-1\right)\left(\frac{\lambda}{(\lambda+\beta)t}\mathbb{I}_{[0,t]}(s) + \frac{\beta}{(\lambda+\beta)t}\mathbb{I}_{[-t,0]}(s)\right)\mathrm{d}s}. \tag{4.32}$$

□

Remark 4.4. *From* (4.32) *we see that, for small values of t, the integrated difference of Poisson processes has Gaussian distribution with mean $(\lambda-\beta)t^2/2$ and variance $(\lambda+\beta)t^3/3$.*

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